



Predictions and Probability

The “Investigations” department features children’s hands-on, minds-on mathematics explorations and presents teachers with open-ended investigations to enhance mathematics instruction. These tasks invoke problem solving and reasoning, require communication skills, and connect various mathematical concepts and principles. The ideas presented here have been tested in classroom settings.

A mathematical investigation is—

- multidimensional in content;
- open ended, with several acceptable solutions;
- an exploration requiring a full class period or longer to complete;
- centered on a theme or event; and
- often embedded in a focus or driving question.

In addition, a mathematical investigation involves processes that include—

- researching outside sources;
- collecting data;
- collaborating with peers; and
- using multiple strategies to reach conclusions.

Although this department presents a scripted sequence and set of directions for a mathematical exploration for the purpose of communicating what happened in this particular classroom, *Principles and Standards for School Mathematics* (NCTM 2000) encourages teachers and students to explore multiple approaches and representations when

engaging in mathematical activities. Each investigation will come alive through students’ problem-solving decisions and strategies in the readers’ own classrooms. As a result of their exploration, students will incorporate their reasoning and proof skills as they evaluate their strategies. The use of multiple approaches creates the richness that is so engaging in an investigation; it also helps students find new ways of looking at things and understand different ways of thinking about a problem.

The activities in the Predictions and Probability investigation lead students to discover the difference between experimental and theoretical probability and to develop reasonable prediction strategies from a sampling experiment. Students begin by using sample picks to predict the colors of marbles or cubes in a basket. Then they use the Probability Simulation application on the TI-73 graphing calculator (see **fig. 1** for instructions) to model various configurations that are the outcome of choosing different-colored marbles from a bag. (Alternatively, they may use actual marbles or centimeter cubes.) Students have several experiences of choosing with replacement and then using that data to predict the marble distribution in the bag, leading to a discussion of definitions for both experimental and theoretical probability (see **fig. 2**). After identifying probability situations as either experimental or theoretical, the investigation expands to predicting probabilities from data in circle graphs. This investigation addresses NCTM’s Content Standard for Data Analysis and Probability (NCTM 2000) and culminates with a discussion of strategies for making reasonable predictions of theoretical probability.

The Investigation

Learning goals, rationale, and pedagogical context

Understanding probability and using it appropriately are necessary skills for functioning as an informed citizen. *Principles and Standards* outlines the impor-

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Edited by Jodelle S. W. Magner, magnerjs@buffalostate.edu, who teach mathematics and mathematics education courses at Buffalo State College in Buffalo, NY 14222. “Investigations” highlights classroom-tested multilesson units that develop conceptual understanding of mathematics topics. This material can be reproduced by classroom teachers for use with their own students without requesting permission from the National Council of Teachers of Mathematics (NCTM). Readers are encouraged to submit manuscripts appropriate for this department by accessing tcm.msubmit.net. Manuscripts must include no more than twelve double-spaced typed pages and two reproducible pages of activities.

tance of reasoning with probability: “Instructional programs from prekindergarten through grade 12 should enable all students to understand and apply basic concepts of probability” (NCTM 2000, p. 400). Probability can be determined theoretically or experimentally. This investigation allows students to explore both types of probability through marble-picking experiments and to “encounter the idea that although they cannot determine an individual outcome ... they can predict the frequency of various outcomes” (p. 181). Computer or calculator simulations may help students confront misconceptions or inaccurate intuitions about probability (NCTM 2000; Tarr, Lee, and Rider 2006). This investigation uses the Probability Simulation application on the TI-73 graphing calculator. The application is also available for the TI-83 Plus and TI-84 Plus families of graphing calculators. Free downloads are available at www.education.ti.com. Alternatively, hands-on materials can be used. In the interest of time, those using hands-on materials may want to decrease both the number of marbles in the containers and the number of marbles picked in the various activities. The lessons were field tested with two groups of students in the Buffalo Public Schools: a sixth-grade class at the Mathematics, Science, Technology School and a group of Community School no. 53 sixth-graders attending the High School Ahead Mathematics Academy, a Saturday morning tutoring program.

Objectives of the investigation

Students will—

- differentiate between experimental and theoretical probability;
- understand why experimental probabilities do not always match the theoretical probabilities; and,
- use a variety of strategies to make reasonable predictions of a sample space based on experimental data.

Materials

For all three lessons, each student needs a TI-73 graphing calculator with the Probability Simulation application loaded on it. The teacher needs an overhead TI-73 graphing calculator unit. Alternatively, instead of a calculator, each student could use a container and a total of thirty objects such as marbles or centimeter cubes representing three different colors. Students will work in cooperative groups of three or four in Lessons 1 and 3 and with a partner in Lesson 2.

Lesson 1

For each student—

- one copy of **activity sheet 1**, “Mystery Marbles”

For each group—

- 3 paper lunch bags
- 90 centimeter cubes (30 blue, 30 green, 30 red)

Lesson 2

For each student—

- three or four copies of **activity sheet 2**, “Find My Marbles”
- if not using graphing calculators:
 - 1 paper lunch bag
 - 90 centimeter cubes (30 each of 3 different colors)

Lesson 3

For each student—

- one set of probability sort cards (see **fig. 3**)
- one circle graph handout (see **fig. 4**)
- one recording sheet (see **fig. 5**)
- if not using graphing calculators:
 - 1 paper lunch bag with 20 centimeter cubes distributed among 5 different colors (for the teacher only)

Previous knowledge

The students who did this investigation were familiar with calculating simple probability and with the concept of a sample space. They also had experience making and justifying predictions.

Lesson 1: Mystery Bags

In the first session, students predict—for each of three different bags—the number of cubes of each color in the bag. Before the class starts, assemble the following three bags of cubes for each group of students (do not let the students see the cubes):

- Bag One—2 blue, 2 green, 6 red cubes;
- Bag Two—1 blue, 6 green, 3 red cubes; and
- Bag Three—7 blue, 2 green, 1 red cube(s).

In Scenario 1, ask students to make a prediction without any data (i.e., without drawing any cubes from the container): “Bag One contains ten cubes. Some are blue; some are green; some are red. How

many cubes of each color are in the bag? Make a prediction without looking or reaching into the bag. Explain how you got your answer.”

Without additional information, many students assumed the number of each color must be roughly the same. Although the actual assignment of colors differed, exactly half of the students predicted a distribution of 4–3–3 and another 42 percent predicted 4–4–2. Only 8 percent predicted a different distribution. Damon’s response indicates how strongly some students hold the misconception of the colors being equally distributed. He wrote, “3 red, 3 blue, 3 green; 1 may be of any color.” As expected, most of the students explained that they just guessed. Lindsey provided a detailed rationale for the 4–3–3 answer: “Because if you divide ten by three, you get 3.33 ... so I took out the decimal.” She then indicated she could add the remaining one to any of the three colors. The students were surprised to find out there were two blue cubes, two green cubes, and six red cubes; many students commented that they had not made good guesses. Explorations with Bag Two and Bag Three will counteract this mindset.

In Scenario 2, ask the students to make a prediction, but reveal all of the cubes first (they need not predict, only report, the color distribution): “Bag Two contains ten cubes. Some are blue; some are green; some are red. Take turns picking a cube and displaying it on your desk. After all ten cubes are on the desk, tell me how many of each color are in the bag and explain how you got your answer.”

Ask the students to discuss how they knew the answer and how this scenario differs from the first one. It is important that students understand the difference between predicting and giving an answer based on a complete set of data.

In Scenario 3, ask students to pick cubes with replacement for a total of ten picks (i.e., the students have experimental data to use as they predict the contents of the bag): “Bag Three contains ten cubes. Some are blue; some are green; some are red. Take turns picking a cube, recording its color, and then putting it back in the bag. Repeat this ten times. You may want to use tally marks to record the picks. After ten picks, tell me how many cubes of each color are in the bag and explain how you got your answer.”

In one class, picking with replacement from Bag Three (seven blue cubes, two green, and one red) produced five blue cubes, four green, and one red. Over half of the students predicted the bag contained five blue cubes, four green, and one red, the exact breakdown of the cubes that were

selected. Most of the remaining students made an adjustment of plus or minus one, predicting six blue cubes, three green, and one red; or four blue cubes, four green, and two red. In the ensuing discussion, most students either had no explanation for their prediction or indicated that they guessed. Only a few connected the picks to their predictions by using explanations such as Nelson’s: “Because she [the teacher] picked more blue [cubes] than anything” or Dyneal’s: “Because we kept on getting blues and greens and only got one red.” Even fewer mentioned the fact that the same cube could have been pulled out more than once. You may want to reinforce the concept that just because the number of picks equals the number of objects in the container, the results from sampling will not necessarily be identical to the contents of the container.

Give each student a copy of **activity sheet 1**. After reading the description of the bag of marbles (ten each of three different colors for a total of thirty marbles), the students should record their individual predictions in the table, explain their thinking, and then discuss their responses with their group members. Then each student should use the Probability Simulation application on the TI-73 graphing calculator to set up a bag of thirty marbles with ten each of three different colors for picking with replacement (see **fig. 1** for instructions). Have each student use the calculator simulation to pick thirty marbles, record the results in the table, and answer the remaining questions on the sheet. The right arrow key will display the number of marbles for each bar in the graph. If your students are not using the calculators, have each group take the cubes from the three bags and redistribute them so that each bag contains cubes of all three colors.

This lesson ends with a whole-class discussion of students’ conclusions. In one class, just over one-third of the students initially predicted that ten of each color would be picked. Davonte reasoned, “Because it is thirty marbles, and $30 \div 3 = 10$,” and Robert reasoned, “Because there are ten of each color.” Another third of the students made predictions close to 10–10–10, such as 8–12–10 or 9–10–11, explaining that they “used probability” to choose numbers close to ten because there were actually ten of each color. Students whose experimental data varied considerably from ten blue, ten green, and ten red were surprised at first. But most of them reassessed their thinking in light of other students’ data from the experiment and the class discussion that followed. During the summary discussion, students should indicate an awareness

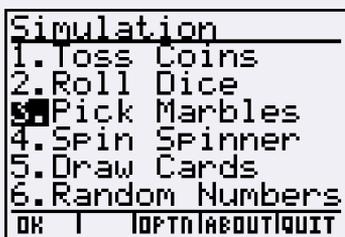
Figure 1

Setting up the application

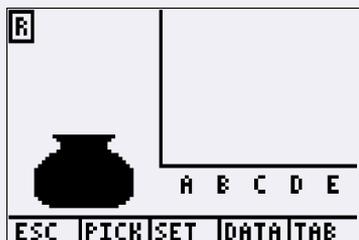
1. Press the *APPS* key and select the *Probability Simulation* application (Prob Sim).



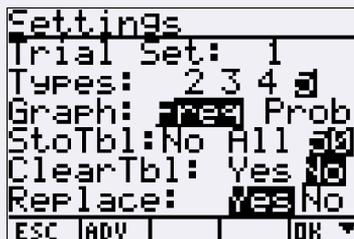
2. From the main menu, choose 3: *Pick Marbles*



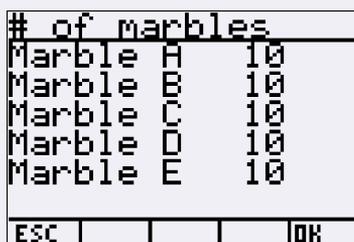
3. Chose *SET* by pressing the (key.



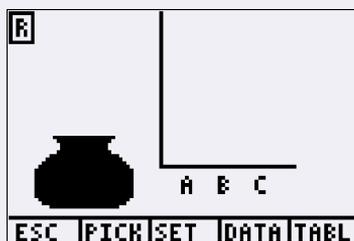
4. You have three marble colors, so change *Types* to 3. Use the cursor to highlight 3, and then press β . Set *ClearTbl* (Clear Table) to *Yes*.



5. Chose *ADV* by pressing the θ key; enter the number of marbles for each color.



6. Press * for *OK* to return to the *Settings* screen. Then Press * for *OK* a second time. If the screen reads, "This will clear all trials from memory. Are you sure you want to continue?" press & to choose *Yes*. You should now be back at the screen showing the bag.



7. Press θ for *PICK* to choose the first marble.

that when sampling with replacement, the same object may be picked multiple times and that some objects may not be picked at all.

Lesson 2: Find My Marbles

In the second part of this investigation, students predict the marble distribution in their partner's bag of marbles after drawing marbles with replacement. Before starting, you may want to have students reflect on what they previously discovered about

picking with replacement and the actual number of marbles in the bag.

Have students create their own bag of marbles according to the guidelines on **activity sheet 2**. Each bag should contain thirty marbles, divided among three colors, and each student may use the color distribution of his or her choice.

Next, students switch bags or calculators with their partners, who then pick thirty marbles (with replacement) and record the picks in the second column of the chart on **activity sheet 2, Part 1**.

Figure 2

Kendall's work

Marble Color	Number of Marbles Picked	Predicted Number in the Bag	Actual Number in the Bag
A	10	1	1
B	63	6	5
C	227	23	24
Total	300	30	30

Students should use their data to predict and record the number of each color of marble in their partner's bag and then explain their reasoning.

Finally, they should check the number of each color that were actually in the bag by looking into the bag or by viewing the settings in the calculator application (see steps 3–5 in **fig. 1**). They should also record the actual numbers and their reactions. Students who finish early should change the number of each color in the bag and repeat the activity. Alternatively, you may want to have the entire class repeat the activity. The students should use a new copy of **activity sheet 2** each time they repeat the activity.

Follow this activity with a whole-class discussion of the students' estimation strategies and their answers to the **activity sheet 2** questions. Generally, the students made predictions that were close to, but for the most part not exactly equal to, the numbers of marbles they had picked. Only a few of their explanations indicated any strategy other than guessing based on their picks. However, some students indicated that, based on their picks, they were confident about which color had either the most or fewest marbles in the bag, although they could not tell the exact number. For example, Dominique wrote, "C had the highest bar [on the graph] by a lot, so I guessed twenty-five out of thirty." Similarly, Leatrice picked 5–12–13 and then reasoned, "I think my partner has [fewer] As than Bs or Cs in his bag." The discussion made it clear that the students needed additional guidance to move toward applying number sense to experimental data in

order to reasonably predict a sample space. So, the students changed the color distribution of the thirty marbles and repeated the activity once more. This time, they were told that although each bag still contained thirty marbles (and **activity sheet 2** tells them to pick thirty marbles), they were not to pick thirty marbles. The hope was that requiring a number of picks that did not equal the number of objects would move them toward the use of proportional reasoning. Many students picked close to thirty times, for example, twenty-five, twenty-nine, or thirty-five times. Unfortunately, those numbers of picks did not easily lend themselves to proportional reasoning. But some chose numbers such as ten or sixty that they thought would be easy either to multiply or divide to reach thirty. A couple of students chose one hundred picks in order to use percentages in their predictions. Kendall made three hundred picks "because it was easy [to compute]." He then divided his experimental data by ten and rounded to the nearest whole number. (His work is shown in **fig. 2**.) One advantage of using technology is the ease and speed of generating such a large number of picks.

To summarize this part of the lesson, have the class repeat the activity a final time while you are the only person picking marbles. This will ensure that the entire class has a common set of data to use as a reference point for the summary discussion. Generate the picks with the overhead calculator from a bag of marbles that you previously set up on your calculator.

Ask the class to reflect on their experiences with this activity and to generate a definition for experimental probability and one for theoretical probability. This class agreed that theoretical probability was “probability found by using the rules.” They described experimental probability as “when you do it and see what happens” and as “hands-on.” After agreeing on your definitions, have the students complete **Part 2** on **activity sheet 2** using the class data from the last repetition of the activity. Conclude the lesson by having a whole-class discussion in which the students compare the experimental and theoretical probabilities and the use of each to predict or identify sample spaces.

Lesson 3: Candy Confusion

In the final section of this investigation, students continue to work with experimental and theoretical probability. Ask each student to sort the Probability Sort Cards (see **fig. 3**) into two groups: situations representing experimental probability and those representing theoretical probability. Have them compare and discuss their results with the others in their group.

To expand the investigation to another representation, give students a circle graph (see **fig. 4**) representing forty total candy bars of five different types. Ask them to individually estimate the theoretical probability of picking each type of candy bar and then to share their strategies and estimated probabilities with their groups. In one class, two students ignored the graph entirely and initially answered one-fifth for each of the five probabilities, maintaining their initial misconception from Lesson 1. However, the rest of the students used visual estimation strategies to create reasonable estimations for the probabilities, with some even drawing in an additional horizontal line segment through the KitKat section in order to “see where the other fourth is.”

Next, ask students how they could find an experimental probability for picking each type of candy bar. As a class, these students decided to use a paper clip held at the center of the circle graph as a spinner, spin forty times, and then use the number of spins in a section divided by forty as their estimate for that section.

The culminating activity for this investigation is for students to use experimental probability to predict how many of each type of candy bar is in a bag containing twenty actual candy bars. Before the start of class, assemble a bag of twenty marbles in five colors to model a candy distribution of your choice. You may use marbles or use the Probability Simulation application set for five colors. Inform

Figure 3

Probability sort cards

Justin flips a coin 10 times
and gets 7 heads.
He says the probability
of a heads is $7/10$.

Susan says the probability of flipping a coin
and getting a heads is $1/2$
because there are
two sides and one is heads.

Maliik reaches into a bag of 5 marbles
and picks out 3 blue marbles.
He says the probability of picking a blue
marble is $3/5$.

Jemario uses the Probability Simulation APP
to pick a marble from a bag containing
20 marbles. He replaces the marble and
picks again. After 17 picks, he says the
probability of picking a red marble is $17/20$.

Mikayla looks at a die
and says the probability of rolling a 3 is $1/6$.

Leatrice rolls a die six times
and gets the following numbers: 4, 5, 3, 4, 1, 1.
She says the probability of rolling a 3 is $1/6$.

Figure 4

A sample circle graph



the students that the distribution in your bag does not match the circle graph in **figure 4**, so you may want them to put away the circle graphs. Give each student a recording chart (see **fig. 5**) to organize their work. Let each student individually decide how many marbles they would like picked from the bag before making their candy bar predictions, with a maximum of five hundred picks. If you are not using a calculator, you will want to choose a smaller maximum number of picks. Display the

actual cubes or the calculator screen on the overhead as you make the picks.

About half the students avoided the use of proportional reasoning by making their predictions after twenty marbles were picked, because “there are twenty in the bag.” About 25 percent used fifteen marbles to predict, indicating that “fifteen is close to twenty.” The remaining students (about 25 percent of the class) picked numbers slightly larger than twenty: either twenty-five, thirty, thirty-five, or forty-five. The students enjoyed watching the marble simulation and wanted to keep going until five hundred marbles had been picked, even though they had all made their predictions based on sixty or fewer picks. I completed the five hundred picks and displayed the results, but there was no evidence that they used the larger number of picks to modify their predictions.

Beyond the Lesson

Students enjoyed using the marble simulation to predict and continued to use it in subsequent classes as time allowed. They changed the difficulty by modifying the number of colors, the number of marbles in the bag, or the number of marbles picked. Also, gathering such experimental data is one method to determine whether or not a game is fair. After a lesson on fair games, some students created their own games based on picking marbles and then used the

Figure 5

Recording sheet

You may choose between 1 and 500 picks. How many picks do you want before you make your prediction? Number of picks _____

Type of Candy	Number Picked	Experimental Probability	Predicted Number	Theoretical Probability
A (Kit Kat)				
B (Milk Chocolate)				
C (Mr. Goodbar)				
D (Peanut Butter Cup)				
E (Special Dark)				
Total			20	

simulation to verify that the games were fair. Some of them also solved a fair game challenge, Challenge 26, retrieved from www.figurethis.org.

Reflections

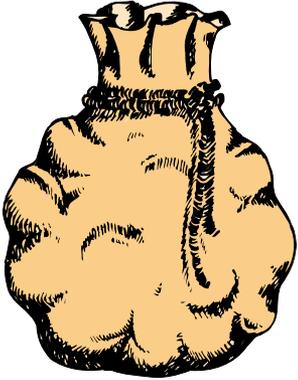
Leading this investigation confirmed to me that some probability concepts are not intuitive to students. For example, in Lesson 3, the students saw no value in picking a large number of marbles before making a prediction, although I offered to pick as many as five hundred marbles using the Probability Simulation application. One advantage of using a calculator simulation is its quick generation of large samples of data. These investigations would have taken significantly longer if the students had picked the same number of marbles by hand for each activity. In addition, the students did not have to tally the picks, because the simulation would display the total number of each color picked at any point.

Probability plays a role in the everyday decisions people make. Students develop their own informal concepts of experimental and theoretical probability outside of the classroom. This investigation

focused on a series of activities that emphasized gathering experimental data and then making reasonable predictions. The activities and discussion involved in this investigation provide opportunities to confront some common but inaccurate ideas about probability and to progress toward a deeper understanding of probabilistic concepts. Whether students use available technology or hands-on materials, “it is useful for students to make predictions and then compare the predictions with actual outcomes” (NCTM 2000, p. 254) to help correct misconceptions about probability.

References

- National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.
- Tarr, James, Hollylynne Stohl Lee, and Robin Rider. “When Data and Chance Collide: Drawing Inferences from Empirical Data.” In *Thinking and Reasoning with Data and Chance*, 2006 Yearbook of the National Council of Teachers of Mathematics (NCTM), edited by Gail Burrill and Portia C. Elliott, pp. 139–49. Reston, VA: NCTM, 2006. ▲

MYSTERY MARBLES

A bag contains a total of thirty marbles.

There are three different colors of marbles:
Color A, Color B, and Color C.

There are ten marbles of each color in the bag.

Suppose you pick a marble, write down its color, and then return it to the bag. Predict how many times a marble of each color will be picked if you do this thirty times. Remember that each time you pick a marble, you return it to the bag before picking again.

Marble Color	Predicted Number	Actual Number Picked
A		
B		
C		
Total		

1. Explain how you arrived at your predictions.
2. Now use bags of marbles or the Probability Simulation APP (application) to pick a marble and return it to the bag. Do this a total of thirty times. Record your results in the table above. You can use the right arrow key to see the number of marbles for each bar in the graph.
3. Do your actual results match your predictions? If not, how are they different?
4. Should your actual results match your predictions? Explain why or why not.

FIND MY MARBLES

Part 1

Assemble a bag of marbles or use the Probability Simulation APP to create a bag of marbles.

Your bag should contain a total of thirty marbles. There are three different colors of marbles. Divide your thirty marbles among the three colors however you like.

Switch calculators with your partner.

Use the Probability Simulation APP to pick thirty marbles from your partner's bag. When you have thirty marbles, fill in your results on the chart to the right. You can use the right arrow key on the calculator to see the number of marbles for each bar in the graph.



Marble Color	Predicted Number	Predicted Number in Bag	Actual Number in Bag
A			
B			
C			
Total			

1. Predict how many marbles of each color your partner has in his or her bag. Explain how you arrived at your predictions. Remember that there are thirty total marbles in the bag.
2. Now check the calculator settings to see how many of each color are in the bag. Are you surprised? Why or why not?

Part 2

Marble Color	Experimental Probability	Theoretical Probability
A		
B		
C		
Total		